1\. (a) Let $f(p) = p^3 + p^2 - 5p - 2$, then $f(2) = 2^3 + 2^2 - 10 - 2 = 0$ *(M1)(C1)*

Therefore $p = 2$ is a solution of $f(p) = 0$. *(C1)*

Now $f(p) = (p-2)(p^2 + 3p + 1) = 0$
(Students may use other approaches, use your discretion)

The rest of the solutions can be found by using the quadratic formula on $p^2 + 3p + 1 = 0$ *(M1)*

Hence these solutions are $p = \dfrac{-3 \pm \sqrt{5}}{2}$ *(A1)*

(b) (i) Arithmetic sequence: 1, $1+p$, $1+2p$, $1+3p$ *(A1)*

Geometric sequence: 1, $p$, $p^2$, $p^3$ *(A1)*

(ii) $(1+2p)+(1+3p) = p^2 + p^3 \Rightarrow p^3 + p^2 - 5p - 2 = 0$ *(C2)*

Therefore, from part (i), $p = 2$, $p = \dfrac{-3 \pm \sqrt{5}}{2}$ *(C1)*

(iii) (a) The sum to infinity of a geometric series exists iff $|p| < 1$, *(C1)*

Hence, $p = \dfrac{-3+\sqrt{5}}{2}$ is the only such number. *(A1)*

(b) The sum to infinity of the geometric series is

$$\frac{a}{1-r} = \frac{1}{1 - \dfrac{\sqrt{5}-3}{2}}$$ *(C2)*

Which converges to $\dfrac{2}{5-\sqrt{5}} = \dfrac{1}{2} + \dfrac{1}{10}\sqrt{5}$ *(A1)*

(c) The sum of the first 20 terms of the arithmetic series can be found by applying the sum formula

$S_{20} = 10(2a + 19d) = 10(2 + 19p)$ *(C1)*

So, $S_{20} = 10\left(2 + 19\left(\dfrac{\sqrt{5}-3}{2}\right)\right) = -265 + 95\sqrt{5}$. *(A1)*

3. (i) (a) The vectors $\bar{l}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$, and $\bar{l}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ are parallel to $L_1$ and $L_2$ respectively. *(C1)*

Also, if we let $\bar{u} = -2\bar{i} + 11\bar{j} + 7\bar{k}$, then
$\bar{u} \cdot \bar{l}_1 = -6 - 22 + 28 = 0$ and $\bar{u} \cdot \bar{l}_2 = -4 + 11 - 7 = 0$ *(C1)*

Therefore the required vector is perpendicular to both lines. *(C1)*

(b) $L_1$ and $L_2$ have the following parametric equations:

$L_1 : x = 3t + 2,\ y = 1 - 2t,\ z = 4t;$ and $L_2 : x = 2u - 2,\ y = u + 1,\ z = 2 - u$ *(C1)*

For an intersection to happen, the following system must be consistent

$$\begin{aligned} 3t + 2 &= 2u - 2 \quad (1) \\ 1 - 2t &= u + 1 \quad (2) \\ 4t &= 2 - u \quad (3) \end{aligned}$$

*(R1)*
*(C1)*

from (2) and (3) we have $t = 1$. Hence $u = -2$. However these values do not satisfy (1) since $5 \neq -6$. Therefore the lines cannot meet. *(A1)* *(R1)*

(c) Since $\bar{u} = -2\bar{i} + 11\bar{j} + 7\bar{k}$, is perpendicular to both $L_1$ and $L_2$, and since the plane contains one of the lines and is parallel to the other, therefore $\bar{u}$ is normal to the plane. The equation of this plane is of the form *(R1)*

$$-2x + 11y + 7z = d.$$

The point (2, 1, 0) lies on $L_1$ and on $P$, therefore *(R1)*

$-4 + 11 + 0 = d$, hence $d = 7$ and the equation is
$-2x + 11y + 7z = 7$. *(A1)*

(ii) (a) The required system is

$$\begin{pmatrix} -1 & k+1 & -1 \\ 1 & 1 & k-2 \\ 2 & 2 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ k+2 \\ k^2 - 2k - 8 \end{pmatrix}$$

*(C2)* (-1 for each error)

(b) $\det(M) = 0 \Rightarrow 2(k-2) - k - (k+1)(k - 2(k-2)) - 0 = 0$
$\Rightarrow k^2 - 2k - 8 = 0$ *(M1)*
$\Rightarrow k = 4$ or $k = -2$ *(A1)*
*(A1)*

(c) (i) Using Gaussian elimination on the following matrix will lead us to the solution:

$$\begin{pmatrix} -1 & k+1 & -1 & 0 \\ 1 & 1 & k-2 & k+2 \\ 2 & 2 & k & k^2-2k-8 \end{pmatrix}$$

*(C1)*

with 2 elementary row operations, the system will be

$$\begin{pmatrix} -1 & k+1 & -1 & 0 \\ 0 & k+2 & k-3 & k+2 \\ 0 & 0 & -k+4 & k^2-4k-12 \end{pmatrix}$$

*(C1)*

Therefore the value of $z$ can be read from the last row

$$z = \frac{k^2-4k-12}{4-k}$$

*(A1)*

(ii) When $k = 4$, the value of $z$ is undefined, and hence the system has no solution. *(R1)*

When $k = -2$, the system is a homogenous system, and hence it admits at least one solution. It is reduced to $x + y = 0$ *(R1)*

$$z = 0$$

Hence it has an infinite number of solutions. *(A1)*

For all other values of $k$, the system has a unique solution. *(A1)*

4. (i) (a) $y = 2x\sin x + \cos 2x$

$$\frac{dy}{dx} = 2\sin 2x + 4x\cos 2x - 2\sin 2x = 4x\cos 2x$$ *(M2)(AG)*

(b) Required volume $V = \pi\int_0^{\pi/2}\left(\sqrt{x}\cos x\right)^2 dx = \pi\int_0^{\pi/2}\left(x\cos^2 x\right) dx$ *(C1)*

$$= \frac{\pi}{2}\int_0^{\pi/2} x(1+\cos 2x)\,dx = \frac{\pi}{2}\int_0^{\pi/2}(x + x\cos 2x)\,dx$$ *(C1)*

and now using integration by parts, or otherwise, on $\int_0^{\pi/2} x\cos 2x\,dx$ will yield

$$V = \frac{\pi}{2}\left(\frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)_0^{\pi/2}$$ *(M1)(A1)*

$$= \frac{\pi}{2}\left(\frac{\pi^2}{8} - \frac{1}{4} - \frac{1}{4}\right) = \frac{\pi}{16}\left(\pi^2 - 4\right)$$ *(A1)(AG)*

(ii) (a) $g(x) = x + 1 - e^x \Rightarrow g'(x) = 1 - e^x = 0$ when $x = 0$,
also, $g'(x) < 0$ when $x > 0$ and $g'(x) > 0$ when $x < 0$; *(M1)(A1)*

$\therefore g(x)$ will increase for $x < 0$ and decrease for $x > 0$, hence the function has a maximum value at $x = 0$, $g(0) = 0$.

$\therefore g(x) \le 0\ \forall\ x \in \mathbb{R}, \Rightarrow x + 1 - e^x \le 0, \Rightarrow x + 1 \le e^x$ *(R2)(A1)*

*(R1)*

(b)(i) From the previous result, we have $e^x - 1 \ge x$, and since it is given that $x > 0$, and we know that $e^x > 0$ dividing both sides of this inequality by $xe^{2x}$ will yield

$$\frac{e^x - 1}{xe^{2x}} > \frac{1}{e^{2x}}, \text{ that is } f(x) > e^{-2x}.$$ *(R1)*

*(C1)*

(ii) $f(x) = \dfrac{e^x - 1}{xe^{2x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \dfrac{1 - e^{-x}}{xe^x}$

*(R1)*

(iii) also from (ii)(a) and since $g(x) \le 0$ for all $x \in \mathbb{R}$, and since replacing $x$ by $-x$ will result in reflecting the function about the $y$-axis, therefore its maximum will not change, and hence $g(-x) \le 0$.

*(R2)*

$g(-x) = -x + 1 - e^{-x} \le 0 \Rightarrow 1 - e^{-x} \le x$, and dividing both sides of this inequality by the positive expression $xe^x$ will yield $f(x) < e^{-x}$.

*(R1)*

(iv) The two results above give $e^{-2x} \le f(x) \le e^{-x}$, and since

$\lim\limits_{x\to 0^+} e^{-2x} = 1$, and $\lim\limits_{x\to 0^+} e^{-x} = 1$

then as $x$ approaches zero from the right $1 \le f(x) \le 1$, therefore *(C1)*

$\lim\limits_{x\to 0^+} f(x) = 1$

*(R1)(A1)*

5. (i) (a) To prove that a set is a group under an operation, we need to prove that:

(i) The set is closed under multiplication:

consider any two elements $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ in $S$. *(C1)*

Now $x \cdot y = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$.

Also *(C1)(A1)*

$(ae+bg)(cf+dh)-(ce+dg)(af+bh) =$

$aecf + aedh + bgcf + bgdh - aecf - bceh - adgf - bgdh =$

$ad(eh-gf)+bc(gf-eh) = (eh-gf)(ad-bc) = 1.$ *(M1)*

Therefore $x \cdot y \in S$. *(C1)*

(ii) The operation is associative. This is true for multiplication of matrices, hence true for this case. *(R1)(C1)*

(iii) There is an identity element. Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is such that $ad - bc = 1,$ hence it belongs to $S$. *(R1)(C1)*

(iv) Every element has an inverse. Since for every $x \in S$, $ad - bc = \det(x) \neq 0,$ therefore it is invertible. *(R1)(C1)*

(b)(i) To prove that $G$ is a subgroup of $S$, it is enough to prove that it is non-empty, closed, and that it contains the inverse of each of its elements. *(R1)*

Clearly $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G,$ hence the first condition is satisfied. *(C1)*

The elements of $G$ are of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, also,

$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} e & f \\ -f & e \end{pmatrix} = \begin{pmatrix} ae-bf & af+bh \\ -bh-af & ae-bf \end{pmatrix} \in G$, and the second *(R2)*

condition is satisfied. Also $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in G,$ *(R1)(C1)*

Hence $G$ is a subgroup of $S$.

(ii) $k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and since $k^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $k^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *(M2)*

therefore the order of $k$ is 4. *(A1)*

(iii) The elements of $H$ are $k, k^2,$ and $k^4$ which were found in part (b) above. The fourth element is $k^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. *(C2)*

(iv) The multiplication table for $H$ is

| | $k^4$ | $k$ | $k^2$ | $k^3$ |
|---|---|---|---|---|
| $k^4$ | $k^4$ | $k$ | $k^2$ | $k^3$ |
| $k$ | $k$ | $k^2$ | $k^3$ | $k^4$ |
| $k^2$ | $k^2$ | $k^3$ | $k^4$ | $k$ |
| $k^3$ | $k^3$ | $k^4$ | $k$ | $k^2$ |

*(C2)* (-1 for each error)

(c) (i) By setting up a multiplication table, we can verify that the set is a group.

| | 1 | −1 | i | −i |
|---|---|---|---|---|
| 1 | 1 | −1 | i | −i |
| −1 | −1 | 1 | −i | i |
| i | i | −i | −1 | 1 |
| −i | −i | i | 1 | −1 |

The set is closed under the operation since the product of any two elements is another element of the set. Also there is an identity element 1, and every element has an inverse since 1 appears in every row once. The operation is associative. *(R3)*

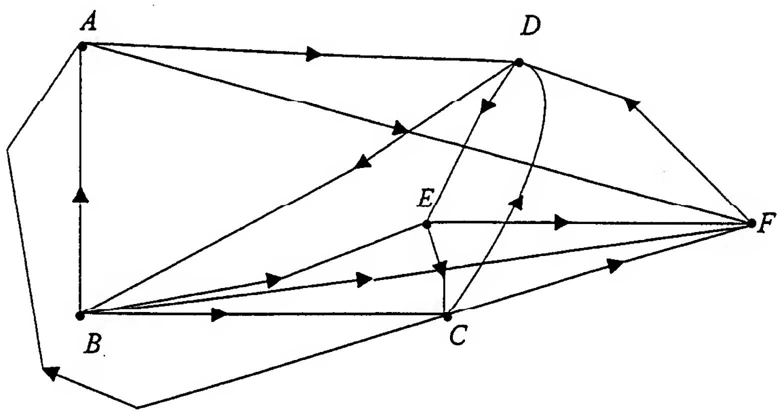
Since multiplication of complex numbers is a commutative operation, the group is Abelian. *(C1)*

(ii) Two groups $G$ and $H$ are isomorphic if there is a one to one correspondence from $G$ to $H$ which preserve the group operation. *(C1)*
The correspondence, $1 \leftrightarrow k^4, -1 \leftrightarrow k^2, \mathrm{i} \leftrightarrow k^3, -\mathrm{i} \leftrightarrow k$ is such a 1-1 correspondence.
Hence $H$ and $K$ are isomorphic. *(C2)*
Since the two groups are isomorphic, and $K$ is Abelian, then $H$ has to be Abelian. *(R1)*

(ii) A cyclic group is a group $G$ with at least one element $a$, such that every element $b \in G$ can be expressed as $b = a^{\mathrm{i}}$ for some $\mathrm{i} < n$, and $a^n = \mathrm{e}$ where $n$ is the order of the group. *(C2)*
Clearly $a^0 = 1$, and $a$ generates the group elements. Also $a^{n+1} = \mathrm{e}^{2\pi \mathrm{i}} = 1$, hence the order is $n + 1$. *(R2)*

(iii) For the group to be Abelian, $ab = ba$ for all $a$ and $b$ in $G$.
Now, $(ab)^2 = (ab)(ab)$, and $(ab)^2 = a^2b^2$, therefore, $abab = aabb$, *(C1)*
multiplying on the left by $a^{-1}$ and on the right by $b^{-1}$ will yield $ab = ba$. Hence, the group is Abelian. *(M2)* *(R1)*

6. (i) (a) Any layout is acceptable.




(C4)

(-1 for each error)

(b) An isomorphism is a $1-1$ correspondence so that images of any two adjacent vertices are also adjacent and conversely. By looking at the adjacency matrix of $G$, we observe that $F$ has an out degree of 1 and so does $X$ in $H$, we also observe that $B$ has an out degree of 4 and so does $Z$. Therefore these should be images of each other in the required isomorphism. We also see that $C$ has an out degree of 3 and so does $U$ in $H$. $D$ has a 2 out and 3 in degrees, so does $W$. By similar reasoning, we have $A$ and $V$ correspond as well as $E$ and $Y$. Hence the required isomorphism is:
$A \leftrightarrow V, B \leftrightarrow Z, C \leftrightarrow U, D \leftrightarrow W, E \leftrightarrow Y,$ and $F \leftrightarrow X$.

(C1) (R1) (R2) (C1)

(c) For a simple connected graph $G$, with at least three edges, to be planar, then according to Euler's formula $e \leq 3v-6$
but $14 > 18-6$, therefore $G$ cannot be planar.

(R1)(C1)
(R1)(C1)

(ii) (a) Prim's algorithm requires that we start at any vertex at random and consider it as a tree, then look for the shortest path that joins a vertex on this tree to any of the remaining vertices and add it, then repeat the last step until all the vertices are on the tree.

(R2)

Starting at $F$, *(other forms can be used)*

1. The shortest path from $F$ to $ABCDEGH$ is to $E$, add $FE \Rightarrow$ weight added $= 90$
2. The shortest path from $FE$ to $ABCDGH$ is to $G$, add $FG$ with weight 120,
3. The shortest path from $GFE$ to $ABCDH$ is to $H$, add $GH$ with weight 220,
4. The shortest path from $GFEH$ to $ABCD$ is to $C$, add $FC$ with weight 230,
5. *(ABD)* Add $CB$ with weight 240,
6. *(AD)* Add $BA$ with weight 180,
7. *(D)* Add $AD$ with weight 200.

(R5)
(-1 for each error)

Therefore a minimal tree has a weight of 1280.

A1

A sketch should have been included.

(b) To find the minimum path from $A$ to $H$ using Dijkstra's algorithm we assign a weight of zero to the first vertex and a weight of $\infty$ to all vertices that are not in the path. Then we start adding vertices adjacent to the present ones that have a minimum weight. At each iteration, the length of the path to a given vertex is evaluated and the minimum is assigned. *(A1)* *(R2)*

$A(0)$
$AB = 180, \Rightarrow B(180)$
$AD = 200, \Rightarrow D(200)$
$AC = 270, \Rightarrow C(270)$

$ADF = 480$ is the minimum path to $F, \Rightarrow F(480)$
$ADE = 480$ is the minimum path to $E, \Rightarrow E(480)$
$ACG = 620$ is the minimum path to $G, \Rightarrow G(620)$

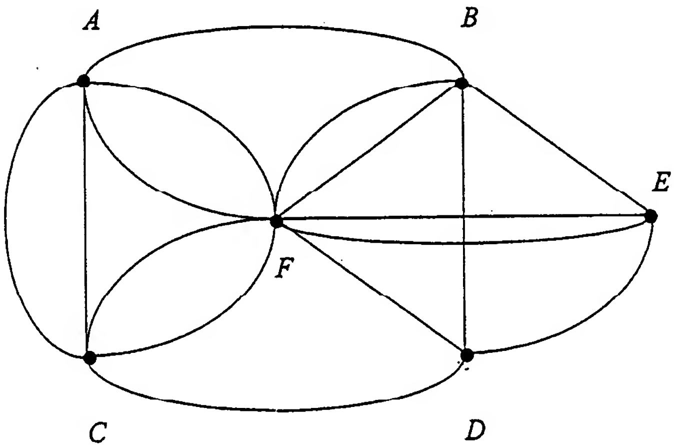
$ADFG = 600$ is the minimum to $G$, $\Rightarrow$ New $G(600)$
$ADEH = 840$ is the minimum to $H, \Rightarrow H(840)$

$ADFGH = 820$ is the minimum to $H$, $\Rightarrow$ NEW $H(820)$.
This is a minimum path from $A$ to $H$.

*(R5)* (-1 for each error)

*(A1)*

(iii) (a)




*(C3)*

(b) There is no such walk since the graph has 4 vertices with odd degree. Therefore there is no Euler's cycle nor a path. *(R1)* *(C1)*

If the door between $A$ and $B$ is locked then there will be a Euler path through the whole graph. There will be only two vertices, $C$ and $F$ with an odd degree. *(R1)*

Therefore our walk should start at one end at the other. Such a walk could be the following: *(R1)*

*FBFEFDBEDCAFACFC.* *(A2)*

(c) Since the degree of each vertex is larger than $\frac{n}{2} = 3$, there is a Hamiltonian circuit. Such a circuit could be *FABEDCF*. *(R1)* *(A1)*

7. (a) The confidence interval gives a range of values which has a 95% probability of containing the mean of the population.

That is the interval $\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} - 1.96\frac{\sigma}{\sqrt{n}}\right)$ if calculated repeatedly, then 95% of such intervals will contain the population mean. 1.96 is the number of standard deviations away from the mean and is read from the standard normal table. *(R2)*

According to the scheme above, the confidence intervals are:

Male: $174 \pm 1.96\left(\frac{12}{20}\right) = (172.824, 175.176)$ *(C1)(A1)*

Female: $173 \pm 1.96\left(\frac{11}{10}\right) = (170.844, 175.156)$ *(C1)(A1)*

Not enough evidence of any difference between the performance of males and females since each of the confidence intervals contain the mean of the opposite sample. *(R2)*

(b) Finding the confidence interval for the difference between the means is similar to the process above, however the standard error is different. The confidence interval is

$$(\mu_M - \mu_F) = \bar{x}_M - \bar{x}_F \pm 1.96\sqrt{\frac{s_M^2}{n_M} + \frac{s_F^2}{n_F}}$$ *(R1)(C1)*

$$= 174 - 173 \pm 1.96\sqrt{\frac{12^2}{400} + \frac{11^2}{100}} = (-1.456, 3.456)$$ *(C1)(A1)*

The result here confirms our previous findings, since this interval contains zero. Which indicates that the situation where the two means are equal is also a member of the confidence interval. *(R1)(C1)*

(c) 1. Form the null hypothesis: $H_0: \mu_M - \mu_F = 0$
The alternate hypothesis: $H_a: \mu_M - \mu_F > 0$ *(C2)*

2. The test statistic is

$$Z_t = \frac{\bar{x}_M - \bar{x}_F - (\mu_M - \mu_F)}{\sqrt{\frac{s_M^2}{n_M} + \frac{s_F^2}{n_F}}} = \frac{174 - 173 - 0}{\sqrt{\frac{144}{400} + \frac{121}{100}}} = 0.798$$ *(C2)(A1)*

3. Critical value for $Z_c$ for a one-tail test is 1.645. *(C1)*
Since $Z_t = 0.798 < Z_c = 1.645$ - it is non-rejection region - we do not have enough evidence that males out perform females on mathematics exams. *(R1)* *(A1)*

(d) This is a matched-pairs difference test since the samples are not independent. We study the differences between pairs and we test the differences with a t-distribution since the sample size is small and we do not know the standard deviation of the population. *(R1)* *(R1)*
The number of degrees of freedom is 9. *(C1)*

The differences are: $1, 2, -2, -1, 0, -2, -1, 1, 1, -2$. Hence the average difference is $\bar{d} = -0.3$ and standard deviation $s_d = 1.494$. *(A2)*

1. $H_0: \bar{d} = 0$
   $H_1: \bar{d} \neq 0$ *(C1)*

2. Test statistic $T = \dfrac{-0.3 - 0}{1.494 / \sqrt{10}} = -0.635$ *(C1)*

3. The critical value with 9 d.f. is $\pm 2.262$. *(C1)*
   Since $-2.262 < T < 2.262$, there is not enough evidence of any difference in speed between the two groups. *(A1)*

(e) This is a goodness of fit test. If the distribution were normal, then the expected frequencies should not be very different from the observed ones. *(R1)*
Using the normal distribution with mean 174 and standard deviation 12, the probabilities corresponding to the different classes are 0.0228, 0.0989, 0.248, 0.322, 0.217 and 0.0912 respectively. Therefore the corresponding expected frequencies are 9, 40, 99, 129, 87 and 36 respectively. *(R1)* *(A1)*

$H_0$: distribution is normal with mean 174 and standard deviation 12.
$H_1$: distribution is not normal with mean 174 and standard deviation 12. *(R1)*

Now $\chi_t^2 = \sum \dfrac{(f_0 - f_e)^2}{f_e} = \dfrac{1}{9} + \dfrac{0}{40} + \dfrac{9}{99} + \dfrac{81}{129} + \dfrac{81}{87} + \dfrac{16}{36}$, *(C2)*
that is $\chi_t^2 = 2.21$. *(A1)*

But $\chi_c^2 = 11.07$ (5 d.f., 5% upper tail). And since the test statistic does not lie in the rejection region, we do not have enough evidence to reject the hypothesis that the distribution is normal with mean 174 and standard deviation 12. *(R2)*

Students using the continuity correction will get:
$\chi^2 = 2.75$.
Conclusion does not change.

8. (i) (a) The ratio test is appropriate here. The ratio of consecutive terms in the series is $r(n) = \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{36}4^n}{n^{36}4^{n+1}} = \dfrac{1}{4}\left(\dfrac{n+1}{n}\right)^{36}$, and hence $\lim\limits_{n\to\infty} r(n) = \dfrac{1}{4}\times 1 = \dfrac{1}{4} < 1$. Thus the series converges by the ratio test. *(C1)* *(M2)(A1)* *(R1)(A1)*

(b) The ratio of consecutive terms in the series is *(C1)*

$$r(n) = \frac{a_{n+1}}{a_n} = \frac{(3n+3)!100^n n!}{(3n)!100^{n+1}(n+1)!}$$

$$= \frac{1}{100}\times\frac{1}{n+1}\times(3n+3)(3n+2)(3n+1) \qquad (M2)$$

$$= \frac{3}{100}\times(3n+2)(3n+1) \qquad (A1)$$

and as $n\to\infty, r(n)\to\infty$, and so the series diverges. *(R1)(C1)*

(ii) (a) Using the trapezium rule, the area under the curve is approximated by

$$T_n = \frac{b-a}{2n}\left[f(x_0)+f(x_n)+2\{f(x_1)+\ldots+f(x_{n-1})\}\right] \qquad (R2)$$

$$= \frac{1}{20}\left[0+0.841+2\left\{\begin{matrix}0.100+0.199+0.296+0.297+0.479+\\ 0.565+0.644+0.717+0.783\end{matrix}\right\}\right] \qquad (M2)$$

$$T_n = 0.450 \qquad (A1)$$

(b) The volume in question is $V = \pi\int_0^1 [f(x)]^2\,dx$, hence the function values in the table have to be squared. *(R2)*

$$V \approx \left\{\frac{b-a}{3n}\right\}[y_0+4y_1+2y_2+\ldots+4y_{n-1}+y_n]\times\pi \qquad (C2)$$

$$V \approx \frac{\pi}{30}\left[\begin{matrix}0^2+4\times0.100^2+2\times0.199^2+4\times0.296^2+\\ 2\times0.297^2+4\times0.479^2+2\times0.565^2+\\ 4\times0.644^2+2\times0.717^2+4\times0.783^2+0.841^2\end{matrix}\right] \qquad (M2)$$

$V \approx 0.268\pi = 0.843$ cubic units. *(A1)*

(iii) (a) The Maclaurin series for a function $f(x)$ is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c)$$ *(R2)*

where $0<|c|<|x|$. Thus,

$$\sin x = 0+x+0\times x^2-\frac{x^3}{3!}+0\times x^4+\frac{x^5}{5!}\cos c,\ 0<|c|<|x|.$$ *(C2)*

(b) In the equation $x^2=\sin x$, replace $\sin x$ by $x-\frac{x^3}{3!}$ to get *(R1)*

$\sin x \approx x-\frac{x^3}{3!}$, or $x^3+6x^2-6x=0$, yields the non - zero roots as

$x=\frac{-6\pm\sqrt{36+24}}{2}=-3\pm\sqrt{15}$. *(C2)*

Obviously $x=-3-\sqrt{15}$ cannot approximate $x^2=\sin x$, *(C1)*

thus $x_0=-3+\sqrt{15}$ is the only approximate solution. *(AG)*

Now

$$|\sin x_0 - x_0^2| = \left|x_0-\frac{x_0^3}{3!}+\frac{x_0^5}{5!}\cos c - x_0^2\right|$$ *(C1)*

$$= \left|x_0-\frac{x_0^3}{3!}-x_0^2+\frac{x_0^5}{5!}\cos c\right|$$

$$= \left|\frac{x_0^5}{5!}\cos c\right|$$ *(C1)*

where $0<c<\sqrt{15}-3$. Since $|\cos x|\le 1$ for all $x$, it follows that

$$|\sin x_0 - x_0^2| \le \left|\frac{x_0^5}{5!}\right| < \frac{\left(\sqrt{15}-3\right)^5}{5!} < \frac{0.9^5}{120} < \frac{1}{200}$$ *(R2)*

Since $\sin x_0 - x_0^2 = \frac{x^5}{5!}\cos c$, where $0<c<\sqrt{15}-3$, and in this interval the cosine function is positive, it follows that $\sin x_0 - x_0^2$ is positive. *(R2)*

(c) $x_1 = x_0 - \frac{\sin x_0 - x_0^2}{\cos x_0 - 2x_0}$, and by setting $x_0=-3+\sqrt{15}$ we obtain *(R1)*

$x_1 = 0.8767$. *(A1)*